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**Future:**.

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***Put-Call-Parity****: Call (K, t ) − Put (K, t ) = PV (F0,t − K ).*

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**Exotic option:**

Cash-or-nothing Call: pays $1 if *ST* >*K* and zero otherwise



Put:



Asset-or-nothing Call: pays *S* (one unit share) if *ST* >*K* and zero



Put









公式汇总（按章节顺序从20章开始）

Claim on

Martingale 公式

本位转换

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则lnx~



**Options on Currencies**



Options on futures



**TUT 11.8**

**1.** Suppose S0=$100, K=$50, r=7.696% (continuously compounded),= 0, and *T* = 1.a. Suppose that for *h* = 1, we have *u* = 1.2 and *d* = 1.05. What is the binomial option price for a call option that lives one period? Is there any problem with having *d >*1? b. Suppose now that *u* = 1.4 and *d* = 0.6. Before computing the option price, what is your guess about how it will change from your previous answer? Does it change? How do you account for the result? Interpret your answer using put-call parity. c. Now let *u* = 1.4 and *d* = 0.4. How do you think the call option price will change from (a)? How does it change? How do you account for this? Use put-call parity to explain your answer. **ANS.** (a)Δ = 1, *B* = −46.296, *price* = 53.704. It is no problem to have a *d* that is larger than one. The only restriction that we have imposed is that *d* *<* *e*(*r* −*δ*)*h* = *e*(0.07696)1 = 1.08, which is respected. (b)We may expect the option premium to go down drastically because with a *d* equal to 0.6, the option is only slightly in the money in the down state at *t* = 1. However, the potential in the up state is even higher, and it is difficult to see what effect the change in *u* and *d* has on the risk-neutral probability. Let’s have a look at put-call-parity. The key is the put option. A put option with a strike of 50 never pays off, neither in (a) nor in (b) because in (a) the lowest possible stock price is 105, and in (b) it is 60. Therefore, the put option has a value of zero. But then, the put-call-parity reduces to: *C* = *S* − *Ke*−0.07696 = 100 − 50 × 0.926 = 53.704. Clearly, as long as the strike price is inferior to the lowest value the stock price can attain at expiration, the value of the call option is independent of *u* and *d*. Indeed, we can calculate: Δ = 1, *B* = −46.296, *price* = 53.704. (c)Again, we are tempted to think in the wrong direction. You may think that, since the call option can now expire worthless in one state of the world, it must be worth less than in part (b). This is not correct. Let us use put-call-parity to see why. Now, with *d* = 0.4, a stock price of 40 at *t* = 1 is admissible, and the corresponding put option has a positive value because it will pay off in one state of the world. We can use put-call-parity to see that: *C* = *S* − *Ke*−0.07696 + *P* = 100 − 50 × 0.926 + *P* = 53.704 + *P* *>* 53.704. Indeed, we can calculate: Δ = 0.9, *B* = −33.333, *price* = 56.6666. **2.** The price of a share of stock is $65. The stock pays dividends at a continuously compounded rate of 5%. The stock’s volatility is 27%. The price evolution of the stock follows the forward tree with each period being 1 year in length. The continuously compounded risk-free interest rate is 8%. A 1-year European put option on the stock has a strike price of $63. The market price of the put option is $6.00. An arbitrageur constructs a strategy involving the purchase or sale of exactly one of the European put options. Determine the accumulated arbitrage profits at the end of one year. **ANS.** The values of *u* and *d* are *u=*1.34986, *d*=0.78663. The possible stock prices are therefore: 65*u*=87.7409, 65*d*=51.1310. If the stock price moves up, then the option pays $0. If the stock price moves down, then the option pays $11.8692. To replicate the option, the investor must purchase delta shares of the underlying stock: Δ =-0.3048, *B*=26.2587. The purchase of the option can therefore be replicated by selling 0.3084 shares of stock and lending $26.2587. Therefore, the price of the put option should be: *V*=6.2127. But the market price of the option is $6.00. Therefore an arbitrageur can earn an arbitrage profit by purchasing the option for $6.00 and synthetically selling it for $6.2127. The synthetic sale of the option requires that the arbitrageur buy 0.3084 shares of stock and borrow $26.2587.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | ***t* = 1** | |
| **Transaction** | ***t* = 0** | ***S*1 = 87.7409** | ***S*1 = 51.1310** |
| Buy the put option | −6.000 | 0 | 11.8690 |
| Sell the synthetic put option | 6.2127 | 0 | −11.8690 |
| **Total** | 0.2127 | 0 | 0 |

The accumulated arbitrage profits at the end of one year = 0.2127*e*0.08 = 0.2304.

**TUT 11.15**

**1.** Assume *K* = $40, = 30%, *r* = 0.08, *T* = 0.5, and the stock is to pay a single dividend of $2 tomorrow, with no dividends thereafter. (a). Suppose *S* = $50. What is the price of a European call option? Consider an otherwise identical American call. What is its price? (b). Repeat, only suppose *S* = $60. (c). Under what circumstance would you not exercise the option today? **ANS.** (a) To be very exact we would have to discount tomorrow’s dividend. However: *PV* (Div) = 2 × exp(−0.08 × 1/360) = 1.9996 = $2. We can now deduct the cash dividend from the current stock price and enter the new value into the Black-Scholes formula: *S*\* = 50 − 2 = 48. Therefore, *C* = $10.2581. We can calculate the price of the American call. It is the maximum of the price of the European call or the value of immediate exercise today: C(American) = max(*S*(0) − *K* , C(European)) = max(50 − 40, 10.2581) = max(10, 10.2581) = 10.2581 = C(European). It is **not** optimal to exercise the American call option early. (b)Now, *C*= 19.6677. C(American) = max(*S*(0) − *K* , C(European)) = max(60 − 40, 19.6677) = max(20, 19.6677) = 20 *>* C(European). In this case, it is actually optimal to exercise the American call option because the value of immediate exercise is higher than the continuation value (as described by the price of the European call option). (c) It is optimal to exercise the American call option today if the cum dividend stock price less the strike price of the option exceeds the Black-Scholes value of the European option. It is important to remember that only dividend paying stocks entail the possibility of early exercise for American call options. **2.** Assume the Black-Scholes framework. Consider a one-year at-the-money European call option on a stock.

You are given: i. The ratio of the call option price to the stock price is less than 10%. ii. The delta of the call option is 0.6. iii. The continuously compounded dividend yield of the stock is 2%. iv. The continuously compounded risk-free interest rate is 4.94%.Determine the stock’s volatility. **ANS.** The value of delta can be used to determine *d*1: , *d*1=0.28. The formula for *d*1 is used to find a quadratic equation in terms of *σ*: For an at-the-money option, *K* = *S*. , *σ* =0.14 or 0.42. The value of *N*(*d*2) depends on the value of *σ*: When *σ* =0.14, *d*2=0.14, *N*(*d*2)=0.5557, when *σ* =0.42, *d*2=-0.14, *N*(*d*2)=0.4443. We are given that the call price is less than 10% of the stock price.

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, . For the inequality above to be satisfied, it must be the case that: *σ* =0.14.

**TUT 11.29**

**1.** The current price of a stock is $60. The volatility of the stock is 30%. The stock pays dividends at a continuously compounded rate of 6%. The continuously compounded expected return on the stock is 22%. The continuously compounded risk-free rate of return is 6%.

An at-the-money European call option on the stock expires in 6 months. The current price of the call option is $5.10 and the delta of the call option is 0.5277. An investor purchases two at-the-money European call options and one at-the-money European put option. Calculate the continuously compounded expected return of the investor’s portfolio. **ANS.** We can use the put-call parity to determine the value of the put option: . *P*=5.10. The value of the portfolio is: *Vport=*2*C+P*=15.30. The delta of the put option is: . The elasticity of the portfolio is: . The expected return of the portfolio is: =0.4444. **3.** Let *S*(*t*) be the price of a stock at time *t*. Under the physical distribution, the stock price changes according to the following process:

, where *S*(0) = 20 and {*Z*(*t*): *t* ≥ 0} is a standard Brownian motion. Determine a. *E*[max(*S*(4) – 18, 0)], b. *E*[max(21 – *S*(4), 0)], where *E*[.] is the expectation under the physical distribution. **ANS.** (a). By Itô’s Lemma, we have

, or

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By using the conditional expectation, we have . Since . we have ,where we have. By using the results of the conditional expectation, . . As a result,

. (b)By using the conditional expectation, we have . By using the similar techniques as in (a) and with the following definition of *d*1 and *d*2, , . we have

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**TUT Martingale and Exotic option**

**1.** Consider the following processes for two non-dividend paying stocks, Stock X and Stock Q: . *Z*(*t*) is a standard Brownian motion. The continuously compounded risk-free rate of return is 3%. The current price of Stock X is $182, and the current price of Stock Q is $77. An investor buys or sells exactly 1 share of stock X as part of a strategy to earn arbitrage. Calculate the arbitrage profit over the time period with the length of *dt*. **ANS.** The Sharpe ratio of Stock Q is greater than that of Stock X: , 0.3333>0.1818. Since both assets follow GBM, a strategy that involves purchasing shares of Stock Q and selling shares of Stock X. results an arbitrage profit of: . Therefore a strategy that involves selling 1 share of Stock X results in an arbitrage profit of:

(*σXX*)(0.1515*dt*)=3.033*dt*. **2.** Given that *S*(*t*) is a GBM (Geometric Brownian motion) which follows , where *Z*(*t*) is a standard Brownian motion under measure *P*. Find another measure *Q* by specifying the Radon-Nikodym derivative of *Q* with respect to *P*, , such that *S*(*t*) is governed by under the measure *Q*, where is a standard Brownian motion under *Q* and *μ*2 is the new drift rate. **ANS.** Let and consider the Radon Nikodym derivative of *Q* with respect to *P* based on the information up to time *t*: . Under the measure *Q*, the stochastic process is a standard Brownian motion under *Q* by the Girsanov Theorem. It is seen that when we set *η* = –0.4 then . Therefore, *S*(*t*) is governed by under measure *Q*. 3. Let *X*(*t*) and *Y*(*t*) be the price processes of the two stocks. Suppose that *X*(0) = 20 and *Y*(0) = 15. Neither stock pays dividends. Under the risk-neutral measure *Q*, *X*(*t*) and *Y*(*t*) are governed by .

where *Z*1(*t*) and *Z*2(*t*) are two independent standard Brownian motions. a. Find *β*. What does *β* stand for? b. By choosing *X*(*t*) as the numeraire, find the corresponding Radon-Nikodym derivative of *QX* with respect to *Q*, . c. Under *QX*, find *dY*(*t*). **ANS.** (a)Let *δX* and *δY* be the dividend yields of Stock X and Stock Y respectively.Since *δX* = 0, *r* = 0.08.*β* = *r* – *δY* = *r* – 0 = 0.08. For non-dividend paying stock, under the risk-neutral measure, *β* is the risk-free interest rate. (b)Assume the money market account starts at $1 at *t* = 0. Let *M*(*t*) be the value of the money market account at time *t*. . (c) From the Girsanov theorem, we have . are independent standard Brownian motions under *QX*.

So, the dynamic of *Y*(*t*) under *QX* is

. **5.** A **chooser option** (also known as an **as-you-like-it option**) becomes a put or call at the discretion of the owner. For example, consider a chooser on the S&R index for which both the call, with value *C(St* , *K*, *T* − *t)*, and the put, with value *P(St* , *K*, *T* − *t)*, have a strike price of *K*. The index pays no dividends. At the choice date, *t*1, the payoff of the chooser is max[*C(St*1, *K*, *T* − *t*1*)*, *P(St*1, *K*, *T* − *t*1*)*] **a.** If the chooser option and the underlying options expire simultaneously, what ordinary option position is this equivalent to? **b.** Suppose that the chooser must be exercised at *t*1 and that the underlying options expire at *T* . Show that the chooser is equivalent to a call option with strike price *K* and maturity *T* plus

*e*−*δ(T*−*t*1*)* put options with strike price *Ke*−*(r*−*δ)(T*−*t*1*)* and expiration *t*1.

**ANS.** (a) Since all the options will be expiring at the same date *t*1, we have the payoff of the chooser will be

. If , then the payoff is . If , then the payoff is . This is equivalent to a *K*-strike straddle. (b)Using put-call parity at *t*1, the value of the as-you-like-it option at *t*1 will be: . The first term is the value of a call with strike *K* and maturity *T* ; the second term is the payoff from holding put options that expire at *t*1 with strike . **6.** A **collect-on-delivery call** (COD) costs zero initially, with the payoff at expiration being 0 if *S <K*, and *S* − *K* − *P* if *S* ≥ *K*. The problem in valuing the option is to determine *P*, the amount the option-holder pays if the option is in-the-money at expiration. The premium *P* is determined once and for all when the option is created. Let *S* = $100, *K* = $100, *r* = 5%, *σ* = 20%, *T* − *t* = 1 year, and *δ* = 0. **a.** Value a European COD call option with the above inputs. **b.** Compute delta and gamma for a COD option. Consider different stock prices and times to expiration, in particular setting *t* close to *T* . **c.** How hard is it to hedge a COD option? **ANS.** With a premium of *P* paid at maturity if *ST* *>* *K*, the COD will have the same value (which will initially be set to zero) as a regular call minus *P* cash or nothing call options. That is, 0 = *BSCall*(*S*0, *K,* *σ,* *r,T* *,* *δ*) − *P* × *CashCall*(*S*0, *K,* *σ,* *r,T* *,* *δ*). (a) Given the inputs and pricing the above options, *P* must satisfy 0 = 10.45 − *P* (0.5323), which implies *P* = 10.45/0.5323 = 19.632. (b)At *t* = 0, the delta of the COD is 0.637 − 19.632 × 0.01875 = 0.2689 and the gamma of the COD is 0.019 − 19.632 (−0.00033) = 2.55%. (c)As the option approaches maturity, the delta will explode when the option is at the money, making delta hedging difficult.

**Assignment 3**

**10.20** Suppose the S&P 500 futures price is 1000, *σ* = 30%, *r* = 5%, *δ* = 5%, *T* = 1, and *n* = 3. **a.** What are the prices of European calls and puts for *K* = $1000? Why do you find the prices to be equal? **b.** What are the prices of American calls and puts for *K* = $1000? **c.** What are the time-0 replicating portfolios for the European call and put? **ANS.** (a)We have to use the formulas of the textbook to calculate the stock tree and the prices of the options. Remember that while it is possible to calculate a delta, the option price is just the value of B because it does not cost anything to enter into a futures contract. In particular, this yields the following prices: For the European call and put, we have: *premium* = 122.9537. The prices must be equal due to put-call-parity. (b)We can calculate for the American call option: *premium* = 124.3347 and for the American put option: *premium* = 124.3347. (c)We have the following time 0 replicating portfolios: For the European call option: Buy 0.5371 futures contracts. Borrow 122.9537 For the European put option: Sell 0.4141 futures contracts. Borrow 122.9537 **Note**: Different from the option on stock, *B* is *positive* for both call and put options on futures contract. **12.6** Suppose XYZ is a non-dividend-paying stock. Suppose *S* = $100, *σ* = 40%, *δ* = 0, and *r* = 0.06. **a.** What is the price of a 105-strike call option with 1 year to expiration? **b.** What is the 1-year forward price for the stock? **c.** What is the price of a 1-year 105-strike option, where the underlying asset is a futures contract maturing at the same time as the option? **ANS.** (a)Using the Black-Scholes formula, we find a call-price of $16.33. (b) We determine the one-year forward price to be:*F*0,*T* (*S*) = *S* × exp(*r* × *T*) = $100 × exp(0.06 × 1) = $106.1837. (c)As the textbook suggests, we need to set the dividend yield equal to the risk-free rate when using the Black-Scholes formula. Thus:*C*(106.1837, 105, 0.4, 0.06, 1, 0.06) = $16.33. This exercise shows the general result that a European futures option has the same value as the European stock option provided the futures contract has the same expiration as the stock option.

**Assignment 4**

**3.** A lognormally distributed $75 stock has a 12% continuously compounded expected rate of return, a zero dividend yield, and a 25% volatility. Determine the 95% prediction interval for the price of the stock after 6 months. **ANS.** The stock price at the end of 6 months is . where *Z* is the standard normal random variable. Let (*SL*, *SU*) be the 95% prediction interval for the price of the stock after 6 months. , ,

, *SL*=55.4433. Similarly, ,

. *SU*=110.8764. So, the 95% prediction interval for the price of the stock after 6 months is (55.4433, 110.8764). **4.** You are given the following information about a stock: i. The price of the stock is lognormally distributed. ii. The expected return on the stock is *α* = 0.17, and the dividend yield is *δ* = 0.02. iii. The probability that the stock price at time 3 months is greater than *K* is 60%. iv. . where is the risk-neutral probability density of *S*0.25 conditional on *S*0. v. The conditional expectation of the stock price in 3 months, conditional on the stock price being less than *K* is: . Calculate the current price of the stock, *S*0. **ANS.** The partial expectation of the stock price in 3 months conditional on *ST* < *K* is given by: . Since the conditions are mutually exclusive and exhaustive, we can add the two partial expectations to obtain the expected value of the stock price in 3 months: . The expected return on the stock is 17%, so, . **5.** You are given the following information: i*. S*(*t*) is the value of the British pound in U.S. dollars at time *t*. iii. The continuously compounded risk-free interest rate in the U.S. is *r* = 0.06. iv. The continuously compounded risk-free interest rate in Great Britian is *r*\* = 0.09. is the forward price in U.S. dollars per British pound, and *T* is the maturity time of the currency forward contract. Based on Itô’s lemma, Find *dG*(*t*). **ANS.** Since we are given the risk-free rate in both the U.S. and Great Britain, we have: *r* – *r*\* = 0.06 – 0.09 = –0.03. The forward price follows geometric Brownian motion: . The partial derivatives are:

. From Itô’s lemma, we have . **6.** The expression for the price of Stock X is: . The price of Stock Q is governed by the following Itô’s process where {*Z*(*t*)} is a standard Brownian motion. Neither Stock X nor Stock Q pay dividends. Calculate the continuously compounded risk-free rate of return. **ANS.** A lognormal stock price implies that changes in the stock price follow geometric Brownian motion, and vice versa:

. We can use the equation provided in the question to find values for α*X* and σ*X* in the price process for Stock X: .The price process for Stock X is therefore: . From the price process for Stock Q, we observe that: *αQ* = 0.08 and *σQ* = 0.13. Stock X and Stock Q are perfectly correlated since they are driven by the same stochastic process, so they must have the same Sharpe ratio: . **20.8** Suppose that ln*(S)* and ln*(Q)* have correlation *ρ* =−0.3 and that *S(*0*)* = $100, *Q(*0*)* = $100, *r* = 0.06, *σS* = 0.4, and *σQ* = 0.2. Neither stock pays dividends. Use equation (20.38) to find the price today of claims that pay **a.** *SQ* **b.** *S/Q.* **ANS.** (a) By using Propositions 20.3 and 20.4, we have . (b) By using Propositions 20.3 and 20.4, we have

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**Final Exam**

**1.** Assume that the Black-Scholes framework holds. You are given: i. The current price of the stock is 105. ii. The continuously compounded risk-free rate of return is 8%. iii. The expected return of the stock is 13% (i.e., *α* = 13%). iv. The stock pays continuously compounded dividends at a rate of 5%. v. Both the call option and the put option expire in 1 year. vi. Both the call option and the put option have a strike price of 100. vii. The Sharpe ratio of the call option is 20%. Find the volatility of the call option (*σ*call) and the volatility of the put option (*σ*put). **ANS.** Since the Sharpe ratio of the call option is equal to the Sharpe ratio of the stock, we can find *σ*Stock:.

. The values of *d*1 and *d*2 are: . . The value of the call option is

. From the put-call parity, . The deltas of the call option and put option are:

. **2.** The price of a stock is $50. The stock does not pay dividends. The continuously compounded risk-free rate of return is 9%. A $50-strike, 3-month European call option has a price of $3.48. The delta of the call option is 0.5725. The gamma of the call option is 0.0508. A $55-strike, 4-month European call option has a price of $2.05. The delta of the call option is 0.3125. The gamma of the call option is 0.0428. A market-maker writes 100 of the the $50-strike call options and **delta-gamma hedges** the position. After 1 day, the new stock price is $51, the new price of the $50-strike call option is $4.12, and the new price of the $55-strike call option is $2.43. Calculate the overnight profit for the market-maker. **ANS.** The gamma of the position to be hedged is: –100(0.0508) = –5.08. We can solve for the quantity, *Q*, of the $55-strike call option that must be purchased to bring the hedged portfolio’s gamma to zero: Q=118.6916. The delta of the position becomes: -100(0.5725)+ 118.6916(0.3125)=-20.1589. The quantity of underlying stock that must be purchased, *QS*, is the opposite of the delta of the position being hedged: *QS* = 20.1589. The cost of purchasing the $55-strike call options and the stock is offset by the value of the $50-strike call options that are sold. The resulting cost of establishing the delta-gamma hedged position is: 20.1589(50)+ 118.6916(2.05)-100(3.48)=903.2628. The cost of establishing the position earns the risk-free rate of interest, so we treat it as borrowing. After 1 day, the value of the $50-strike options has changed by: −100(4.12 – 3.48) = –64.00. After 1 day, the value of the $55-strike options has changed by: 118.6916(2.43 – 2.05) = 45.1028. After 1 day, the value of the shares of stock has changed by: 20.1589(51 – 50) = 20.1589. After 1 day, the value of the funds that were borrowed at the risk-free rate has changed by:

–903.2628(*e*0.09/365 – 1) = –0.2227. The overnight profit is $1.039.

**5.** Let X(t) and Y(t) be the price processes of the two stocks.

Suppose that X(0) = 100 and Y(0) = 100. Neither stock pays dividends. Under the risk-neutral measure Q, X(t) and Y(t) are governed by

where Z1(t) and Z2(t) are two independent standard Brownian motions.

Consider a European type financial derivative with 2 years to expiration. The payoff at the maturiy is given by

where is an indicator function and is defined as



Find the price of the financial derivative at time 0.





